17 Source Problems for Heat and Wave IBVPs

We have mostly dealt with homogeneous equations, homogeneous b.c.s in this course so far. Recall that if we have non-homogeneous b.c.s, then we want to first make a transformation of the problem to one that has homogeneous b.c.s since we can only solve the EVP arising from separation-of-variables method if the b.c.s are homogeneous. The transformation may lead to a new problem with homogeneous b.c.s but with a non-homogeneous equation. Of course, there are modeling cases where the equation is naturally non-homogeneous for physical reasons, like the presence of sources or sinks. So now we have to discuss that possibility.

First consider the case where the non-homogeneous term sends on x only:

$$\begin{cases} u_t = u_{xx} + F(x) & 0 < x < 1 \ , \ t > 0 \\ u(0,t) = 0 = u(1,t) & t > 0 \\ u(x,0) = f(x) & 0 < x < 1 \end{cases} \tag{1}$$

Note that a **steady state** solution to this problem, $u(x,t) = U(x)$, would be the solution to

$$0 = \frac{d^2U}{dx^2} + F(x)$$
$$0 = U(0) = U(1)$$

Therefore, by integrating twice, the solution to this boundary-value problem (BVP) is

$$U(x) = x \int_0^1 \int_0^y F(z) dz dy - \int_0^x \int_0^y F(z) dz dy \ .$$

Exercise: Show this.

Hence, if we write the solution to problem (1) as $u(x,t) = U(x) + w(x,t)$, then $w(x,t)$ satisfies the problem

$$\begin{cases} w_t = w_{xx} & 0 < x < 1 \ , \ t > 0 \\ \\ w(0,t) = 0 = w(1,t) & t > 0 \\ \\ w(x,0) = f^*(x) := f(x) - U(x) & 0 < x < 1 \end{cases} \tag{2}$$

which we solve by the usual separation-of-variables method.

To summarize, if you have a problem with non-homogeneous boundary conditions, transform it to one with homogeneous boundary conditions. This may or may not lead to an equation that is non-homogeneous. If there is a non-homogeneous term, and it is just a function of the space variable (x in the 1D case, of vector $\mathbf{x}$ in the multidimensional case), then look for the steady-state solution. After that obtain the homogeneous problem as we did in the example above, and proceed to derive the eigenfunction expansion of the solution to the homogeneous problem.

For the more general problem of a diffusion or wave equation problem with a non-homogeneous term depending on both x and t, we just modify our usual procedure a bit.

Example: Consider

$$\begin{cases} u_t = u_{xx} + F(x,t) & 0 < x < 1 \ , \ t > 0 \\ \\ u(0,t) = 0 = u(1,t) & t > 0 \\ \\ u(x,0) = g(x) & 0 < x < 1 \end{cases}$$

For each $t > 0$, we consider F to be piecewise smooth function in x. For whatever b.c.s are imposed, in this example the Dirichlet b.c.s, consider the analogous *homogeneous* equation with the same b.c.s, and proceed to solve the EVP. (For this example we know the eigenvalues and associated eigenfunctions are given by $\lambda_n = n^2\pi^2$, and $\phi_n(x) = \sin(n\pi x)$, $n \geq 1$). Now **do not** solve the T(t)-equation, but instead write the more general eigenfunction expansion (coefficients being functions of t to be determined), that is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x) \ \text{ with } \ F(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x) \tag{3}$$

Now substitute these series into the original equation:

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin(n\pi x) = -\sum_{n=1}^{\infty} n^2\pi^2 b_n \sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x) \quad ,$$

or

$$\sum_{n=1}^{\infty} \{\frac{db_n}{dt} + n^2\pi^2 b_n\} \sin(n\pi x) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x) \quad .$$

Since Fourier series are unique for a given function, the coefficients must be equal; this means in this case that $\frac{db_n}{dt} + n^2\pi^2 b_n = f_n(t)$. Also, by setting $t = 0$, we have $g(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n(0) \sin(n\pi x)$, that is, the $b_n(0)'s$ are the Fourier sine coefficients for the function $g(x)$. Therefore, if we write $\beta_n := 2\int_0^1 g(x) \sin(n\pi x) dx$, then the problems we must solve for the coefficients for u are

$$\frac{db_n}{dt} + n^2\pi^2 b_n = f_n(t)$$
$$b_n(0) = \beta_n$$

Hence,

$$\frac{d}{dt}(e^{n^2\pi^2 t} b_n) = f_n(t) e^{n^2\pi^2 t} \quad \Rightarrow \quad b_n(t) = \beta_n e^{-n^2\pi^2 t} + \int_0^t e^{-n^2\pi^2(t-s)} f_n(s) ds \ .$$

Of course, we need to know the $f_n's$. They are the Fourier coefficients of $F(x, t)$, and gotten in the usual way of multiplying the $F(x, t)$ expansion in (3) by an arbitrary eigenfunction, and integrating. Because of orthogonality, we have

$$f_n(t) = 2\int_0^1 F(x, t) \sin(n\pi x) dx \ .$$

To summarize, given a problem for $u(x, t)$ with a non-homogenous equation, and homogeneous boundary conditions,

- Use separation-of-variables method on the homogenous equation and boundary conditions to derive the EVP;

- Solve the EVP for the eigenvalues and eigenfunctions, and write the general eigenfunction expansion for u and the forcing term (the non-homogeneity function);

- Substitute these expansions into the partial differential equation and the initial conditions and solve the coefficient equations.

Example: Consider

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin(3x) & 0 < x < \pi \ , \ t > 0 \\ \\ u(0,t) = 0 \ , \ u(\pi,t) = 1 & t > 0 \\ \\ u(x,0) = x/\pi & 0 < x < 1 \end{cases}$$

Since we have non-homogeneous b.c.s, we must deal with them. If we let $u(x,t) = x/\pi + v(x,t)$, then v must satisfy the problem

$$\begin{cases} v_t = v_{xx} + e^{-t}\sin(3x) & 0 < x < \pi \ , \ t > 0 \\ \\ v(0,t) = 0 = v(\pi,t) & t > 0 \\ \\ v(x,0) = 0 & 0 < x < 1 \end{cases}$$

Now let $v(x,t) = T(t)\phi(x)$ and substitute into the homogeneous version of the v equation. Then separating variables gives

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 & 0 < x < \pi \\ \\ \phi(0) = 0 = \phi(\pi) \end{cases}$$

Hence, $\lambda = \lambda_n = n^2$, $\phi = \phi_n(x) = \sin(nx)$, for $n = 1, 2, \ldots$. Now we write $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx)$, so $0 = v(x,0) = \sum_{n=1}^{\infty} b_n(0)\sin(nx)$. This implies $b_n(0) = 0$ for all n. With $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx)$ and $F(x,t) = e^{-t}\sin(3x) = \sum_{n=1}^{\infty} f_n(t)\sin(nx)$, then

$$f_n(t) = \begin{cases} e^{-t} & n = 3 \\ 0 & n \neq 3 \end{cases}$$

Finally, substituting the series into the equation gives

$$\sum_{n=1}^{\infty} \{ \frac{db_n}{dt} + n^2 b_n \} \sin(nx) = e^{-t}\sin(3x) = \sum_{n=1}^{\infty} f_n(t)\sin(nx) \ .$$

So for $n \neq 3$ we have $\frac{db_n}{dt} + n^2 b_n = 0$, $b_n(0) = 0$, which implies $b_n(t) \equiv 0$. For $n = 3$, we have $\frac{db_3}{dt} + 9b_3 = e^{-t}$, $b_3(0) = 0$, so $b_3(t) = \frac{1}{8}(e^{-t} - e^{-9t})$. This gives $v(x,t) = \frac{1}{8}(e^{-t} - e^{-9t})\sin(3x)$, so

$$u(x,t) = x/\pi + \frac{1}{8}(e^{-t} - e^{-9t})\sin(3x) \quad .$$

Note: For the arbitrary initial condition $u(x,0) = f(x)$, $v(x,0) = f^*(x) := f(x) - x/\pi$, so $f^*(x) = \sum_{n=1}^{\infty} b_n(0)\sin(nx)$, which gives $b_n(0) = \frac{2}{\pi}\int_0^\pi f^*(y)\sin(ny)dy$. Then for $n \neq 3$, $b_n(t) = b_n(0)e^{-n^2 t}$, while $b_3(t) = b_3(0)e^{-9t} + \frac{1}{8}(e^{-t} - e^{-9t})$.

Exercises: The next two problems are slight modifications of the above example.

1. Solve the following problem, with $m \neq 1$,

$$\begin{cases} v_t = v_{xx} + e^{-t}\sin(mx) & 0 < x < \pi \ , \ t > 0 \\ v(0,t) = 0 = v(\pi,t) & t > 0 \\ v(x,0) = 0 & 0 < x < 1 \end{cases}$$

   and arrive at the solution $v(x,t) = \frac{1}{m^2-1}(e^{-t} - e^{-m^2 t})\sin(mx)$.

2. Show the solution for

$$\begin{cases} v_t = v_{xx} + h(t)\sin(mx) & 0 < x < \pi \ , \ t > 0 \ \ h \text{ is continuous} \\ v(0,t) = 0 = v(\pi,t) & t > 0 \\ v(x,0) = f(x) & 0 < x < 1 \end{cases}$$

   is $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx)$, where $b_n(0) := f_n = \frac{2}{\pi}\int_0^\pi f(y)\sin(ny)dy$, for all $n \geq 1$, $b_m(t) = f_m e^{-m^2 t} + \int_0^t e^{-m^2(t-\tau)}h(\tau)d\tau$, and for $n \neq m$, $b_n(t) = f_n e^{-n^2 t}$.

Let us re-emphasize the procedure on the following forced wave equation problem. Consider

$$\begin{cases} u_{tt} = u_{xx} + xe^{-t} & 0 < x < 1 \ , \ t > 0 \\ \\ u_x(0,t) = 0 = u_x(1,t) & t > 0 \\ \\ u(x,0) = 0 = u_t(x,0) & 0 < x < 1 \end{cases}$$

1. Consider the homogeneous equation and b.c.s; separate variables and find the EVP. In this case we have

$$\tfrac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\tfrac{d\phi}{dx}(0) = 0 = \tfrac{d\phi}{dx}(1)$$

   Solve for the eigenvalues and eigenfunctions; in this case we have $\lambda_n = n^2\pi^2$, $n = 0, 1, 2, \ldots$, and $\phi_n(x) = \cos(n\pi x)$. (Note, in this case, $\lambda = 0$ is an eigenvalue.)

2. Write u as a Fourier series in these eigenfunctions; thus

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos(n\pi x);$$

3. Substitute this series into the non-homogeneous pde and represent the non-homogeneous term by a similar series, that is

$$\frac{1}{2}a_0'' + \sum_{n\geq 1}\{a_n'' + n^2\pi^2 a_n\}\cos(n\pi x) = xe^{-t} = \frac{1}{2}f_0(t) + \sum_{n\geq 1} f_n(t)\cos(n\pi x)$$

4. Solve for the $f_n's$, then write out equations for the $a_n's$:

$$f_0(t) = e^{-t} \ \ , \ \ f_n(t) = \frac{2[(-1)^n - 1]}{n^2\pi^2}e^{-t} \ \ n \geq 1$$

$$\text{so } a_0'' = f_0(t) \quad a_n'' + n^2\pi^2 a_n = f_n(t) \ \ ;$$

5. Solve for the initial data; for this example it is simple:

$$\tfrac{a_0(0)}{2} + \textstyle\sum_{n\geq 1} a_n(0)\cos(n\pi x) = f(x) \equiv 0 \Rightarrow \ a_n(0) = 0 \ \text{ for all } n \geq 0$$

$$\tfrac{a_0'(0)}{2} + \textstyle\sum_{n\geq 1} a_n'(0)\cos(n\pi x) = g(x) \equiv 0 \Rightarrow \ a_n'(0) = 0 \ \text{ for all } n \geq 0 \ ;$$

6. Solve the differential equations with initial conditions for the coefficients:

$$a_0'' = e^{-t}\ ,\ \ a_0(0) = a_0'(0) = 0\ \Rightarrow\ a_0(t) = e^{-t} + t - 1$$

$$a_n'' + n^2\pi^2 a_n = \tfrac{2[(-1)^n-1]}{n^2\pi^2}e^{-t}\ ,\ a_n(0) = a_n'(0) = 0\ \ \Rightarrow$$

$$a_n(t) = C_1\cos(n\pi t) + C_2\sin(n\pi t) + \frac{2[(-1)^n - 1]e^{-t}}{n^2\pi^2(1+n^2\pi^2)}$$

$$= \frac{2[(-1)^n - 1]}{n^2\pi^2(1+n^2\pi^2)}\{e^{-t} + \frac{\sin(n\pi t)}{n\pi} - \cos(n\pi t)\}$$

$$= \begin{cases} 0 & \text{if } n = \text{even} \\ \frac{-4}{n^2\pi^2(1+n^2\pi^2)}\{e^{-t} + \frac{\sin(n\pi t)}{n\pi} - \cos(n\pi t)\} & \text{if } n = \text{odd} \end{cases}$$

7. Now put is all together:

$$u(x,t) = \frac{1}{2}(e^{-t} + t - 1) - \frac{4}{\pi^2}\sum_{n=odd\geq 1}\frac{e^{-t} + \frac{\sin(n\pi t)}{n\pi} - \cos(n\pi t)}{n^2(n^2\pi^2+1)}\cos(n\pi x)$$

Summary: Review the summary paragraph on page 2, and really understand the procedure outlined by the example on pages 6-7.

Exercises: Develop the eigenfunction series expansion representation for the solution to the following problems:

1. Redo the problem on the top of page 6 with the boundary conditions $u(0,t) = 0 = u(1,t)$.

2. $$\begin{cases} u_t = u_{xx} + e^{-5t}\sin(4x) & 0 < x < \pi\ \ ,\ \ t > 0 \\ u(0,t) = 0 = u(\pi,t) & t > 0 \\ u(x,0) = x(\pi - x) & 0 < x < \pi \end{cases}$$

3.

$$\begin{cases} u_t = u_{xx} + 0.1e^{-t}\sin(\frac{\pi x}{2}) & 0 < x < 1 \;\; , \;\; t > 0 \\ u(0,t) = 0 = u_x(1,t) & t > 0 \\ u(x,0) = 0 & 0 < x < 1 \end{cases}$$

(Answer: $u(x,t) = \frac{2}{5\pi^2 - 20}(e^{-t} - e^{-\pi^2 t/4})\sin(\frac{\pi x}{2})$.)